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GENERALIZED CONVOLUTIONS FOR THE INTEGRAL TRANSFORMS OF FOURIER TYPE AND APPLICATIONS [1]

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Abstract

In this paper we provide several new generalized convolutions for the Fourier-cosine and the Fourier-sine transforms and consider some applications. Namely, the linear space $L^1(\mathbb{R}^d)$, equipped with each of the convolution multiplications constructed, becomes a normed ring, and the explicit solution in $L^1(\mathbb{R}^d)$ of the integral equation with the mixed Toeplitz-Hankel kernel is obtained.

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1. Introduction

The Fourier convolution of two functions g and f is defined by the integral

$$(f \underset{F}{*} g)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} g(y) f(x-y) dy.$$

The theory of the convolutions of integral transforms has been developed for a long time and is applied in many fields of mathematics. Historically, Churchill introduced the generalized convolutions of the integral transforms and found their application for solving boundary value problems in 1940

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(see [8, 9]). In 1958, Vilenkin gave a convolution for the integral transform in a specific space of integrable functions (see [29]). Kakichev presented some methods to build generalized convolutions of integral transforms in 1967; he formulated the concept of the generalized convolutions of integral transforms and dealt with convolutions for power series in 1990 (see [14, 15]). Also, in his article [14] he pointed out that generalized convolutions of many known transforms had not been found yet.

In the recent years, many convolutions, generalized convolutions, and poly-convolutions of well-known integral transforms as the Fourier, Hankel, Mellin, Laplace transforms, and their applications have been investigated (see for example, [4, 5, 6, 7, 10, 16, 17, 24, 26, 27, 30]). However, there have not been so many generalized convolutions of the integral transforms of Fourier type, which from our point of view, deserve interest.

Recall the definitions of the Fourier-cosine and Fourier-sine transforms:

$$(T_c f)(x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \cos(xy) f(y) dy; \quad (T_s f)(x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \sin(xy) f(y) dy,$$

where $\cos(xy) := \cos(<x, y>)$, $\sin(xy) := \sin(<x, y>)$.

The main purpose of this paper is to construct some generalized convolutions for the transformations T_c, T_s, and to solve, by their means, integral equations with mixed Toeplitz-Hankel kernel.

The paper consists of three sections and is organized as follows. In Section 2, we find eight new generalized convolutions with weight-function being the function $\cos xh$, or $\sin xh$ for T_c, T_s. We call h the shift in the convolution transform. From the factorization identities of those convolutions, we emphasize on the fact (perhaps interesting): the shift in the left-side moves only into the weight-function in the right-side. This lays in the basis of our solution of convolutional integral equations with different shifts, as equation (3.5).

There are two subsections in Section 3. In Subsection 3.1, we deal with some normed ring structures of the linear space $L^1(\mathbb{R}^d)$. Namely, the space $L^1(\mathbb{R}^d)$, equipped with each of the convolution multiplications obtained in Section 2, becomes a normed ring. In Subsection 3.2, we provide a sufficient and necessary condition for the solvability of an integral equation with the mixed Toeplitz-Hankel kernel having shifts, and obtain its explicit solution via the Hartley transform by using the constructed convolutions. Finally, the advantage of the convolutional approach to the equations as in Subsection 3.2 over that relating to the Fourier transform is discussed.

2. Generalized convolutions

The nice idea of a generalized convolution focuses on the factorization identity. We now remind the concept of convolutions.

Let U_1, U_2, U_3 be linear spaces on the field of scalars $\mathcal{K}$, and let V be a commutative algebra on $\mathcal{K}$. Suppose that $K_1 \in L(U_1, V)$, $K_2 \in L(U_2, V)$, $K_3 \in L(U_3, V)$ are linear operators from U_1, U_2, U_3 to V, respectively. Let δ denote an element in the algebra V.

Definition 2.1. (see [6, 14, 17]) A bilinear map $* : \ U_1 \times U_2 :\longrightarrow U_3$ is called a *convolution* with weight-element δ for K_3, K_1, K_2 (in that order), if the following identity holds: $K_3(*(f,g)) = \delta K_1(f)K_2(g)$, for any $f \in U_1, g \in U_2$. This identity is called *the factorization identity of the convolution.*

The image $*(f,g)$ is denoted by $f \overset{\delta}{\underset{K_3,K_1,K_2}{*}} g$. If δ is the unit of V, we say briefly the convolution for K_3, K_1, K_2. In the case of $U_1 = U_2 = U_3$ and $K_1 = K_2 = K_3$, the convolution is denoted simply by $f \overset{\delta}{\underset{K_1}{*}} g$, and by $f \underset{K_1}{*} g$ if δ is the unit of V. Observe that the factorization identities play a key role in many applications.

In what follows, we consider $U_1 = U_2 = U_3 = L^1(\mathbb{R}^d)$ with the Lebesgue measure, and let V be the algebra of all measurable functions (real or complex) on $\mathbb{R}^d$.

For any given $h \in \mathbb{R}^d$, put $\alpha(x) = \cos xh$, $\beta(x) = \sin xh$. In this section we provide eight new generalized convolutions for T_c, T_s with weight-function $\alpha(x)$, or $\beta(x)$.

Theorem 2.1. *If $f, g \in L^1(\mathbb{R}^d)$, then each of the integral operations* (2.1), (2.2), (2.3), (2.4) *below defines a generalized convolution as:*

$$\begin{aligned}(f \overset{\alpha}{\underset{T_c}{*}} g)(x) := \frac{1}{4(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \big[&f(x-u+h) + f(x-u-h)\\ &+ f(x+u+h) + f(x+u-h)\big] g(u)du, \qquad (2.1)\end{aligned}$$

$$\begin{aligned}(f \overset{\alpha}{\underset{T_c,T_s,T_s}{*}} g)(x) := \frac{1}{4(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \big[&- f(x-u+h) - f(x-u-h)\\ &+ f(x+u+h) + f(x+u-h)\big] g(u)du, \qquad (2.2)\end{aligned}$$

$$\begin{aligned}(f \overset{\beta}{\underset{T_c,T_s,T_c}{*}} g)(x) := \frac{1}{4(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \big[&f(x-u+h) - f(x-u-h)\\ &+ f(x+u+h) - f(x+u-h)\big] g(u)du \qquad (2.3)\end{aligned}$$

$$(f \underset{T_c,T_c,T_s}{\overset{\beta}{*}} g)(x) := \frac{1}{4(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \big[f(x-u+h) - f(x-u-h) \\ - f(x+u+h) + f(x+u-h)\big] g(u)du. \tag{2.4}$$

P r o o f. Let us first prove the convolution (2.1). We have

$$\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |f \underset{T_c}{\overset{\alpha}{*}} g|(x)dx \leq \frac{1}{4(2\pi)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-u+h)||g(u)|dxdu \\ + \frac{1}{4(2\pi)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-u-h)||g(u)|dxdu \\ + \frac{1}{4(2\pi)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x+u+h)||g(u)|dxdu \\ + \frac{1}{4(2\pi)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x+u-h)||g(u)|dxdu \\ = \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\int_{\mathbb{R}^d} |f(x)|dx\right)\left(\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |g(u)|du\right) < +\infty.$$

Therefore, the integral expression (2.1) is a bilinear map from $L^1(\mathbb{R}^d) \times L^1(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$. We now prove the factorization identity. We have

$$\alpha(x)(T_cf)(x)(T_cg)(x) = \frac{\cos xh}{(2\pi)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \cos xu \cos xv f(u)g(v)dudv \\ = \frac{1}{4(2\pi)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Big(\cos x(u+v+h) + \cos x(u-v+h) + \cos x(u+v-h) \\ + \cos x(u-v-h)\Big) f(u)g(v)dudv = \frac{1}{4(2\pi)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \cos xt\Big[f(t-y-h) \\ + f(t+y+h) + f(t-y+h) + f(t+y-h)\Big] g(y)dydt \\ = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \cos xt (f \underset{T_c}{\overset{\alpha}{*}} g)(t)dt = T_c(f \underset{T_c}{\overset{\alpha}{*}} g)(x),$$

as desired. Thus, the convolution (2.1) is proved.

By using the following identities

$$
\begin{aligned}
\cos xh \sin xu \sin xv &= \frac{1}{4}\Big[-\cos x(u+v+h) + \cos x(u-v+h) \\
&\qquad - \cos x(u+v-h) + \cos x(u-v-h)\Big], \\
\sin xh \sin xu \cos xv &= \frac{1}{4}\Big[-\cos x(u+v+h) - \cos x(u-v+h) \\
&\qquad + \cos x(u+v-h) + \cos x(u-v-h)\Big], \\
\sin xh \cos xu \sin xv &= \frac{1}{4}\Big[-\cos x(u+v+h) + \cos x(u-v+h) \\
&\qquad + \cos x(u+v-h) - \cos x(u-v-h)\Big],
\end{aligned}
$$

we can prove the convolutions (2.2), (2.3), (2.4). The theorem is proved. ∎

The following identities hold also:

$$
\begin{aligned}
\cos xh \cos xu \sin xv &= \frac{1}{4}\Big[\sin x(u+v+h) + \sin x(u+v-h) \\
&\qquad - \sin x(u-v+h) - \sin x(u-v-h)\Big], \\
\cos xh \sin xu \cos xv &= \frac{1}{4}\Big[\sin x(u+v+h) + \sin x(u+v-h) \\
&\qquad + \sin x(u-v+h) + \sin x(u-v-h)\Big], \\
\sin xh \sin xu \sin xv &= \frac{1}{4}\Big[\sin x(u-v+h) - \sin x(u-v-h) \\
&\qquad - \sin x(u+v+h) + \sin x(u+v-h)\Big], \\
\sin xh \cos xu \cos xv &= \frac{1}{4}\Big[\sin x(u+v+h) - \sin x(u+v-h) \\
&\qquad + \sin x(u-v+h) - \sin x(u-v-h)\Big].
\end{aligned}
$$

Then, similarly to the proof of Theorem 2.1, we can prove the following theorem.

Theorem 2.2. *If $f, g \in L^1(\mathbb{R}^d)$, then each of the integral transforms (2.5), (2.6), (2.7), (2.8) below defines a generalized convolution as:*

$$
\begin{aligned}
(f \underset{T_s,T_c,T_s}{\overset{\alpha}{*}} g)(x) &:= \frac{1}{4(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \big[f(x-u+h) + f(x-u-h) \\
&\qquad - f(x+u+h) - f(x+u-h)\big] g(u)du, \qquad (2.5)
\end{aligned}
$$

$$\begin{aligned}(f \underset{T_s,T_s,T_c}{\overset{\alpha}{*}} g)(x) &:= \frac{1}{4(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \big[f(x-u+h) + f(x-u-h) \\ &\quad + f(x+u+h) + f(x+u-h)\big] g(u)du, \qquad (2.6)\\ (f \underset{T_s}{\overset{\beta}{*}} g)(x) &:= \frac{1}{4(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \big[f(x-u+h) - f(x-u-h) \\ &\quad - f(x+u+h) + f(x+u-h)\big] g(u)du, \qquad (2.7)\\ (f \underset{T_s,T_c,T_c}{\overset{\beta}{*}} g)(x) &:= \frac{1}{4(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \big[- f(x-u+h) + f(x-u-h) \\ &\quad - f(x+u+h) + f(x+u-h)\big] g(u)du. \qquad (2.8)\end{aligned}$$

Example 2.1. Consider $d = 1$. Put $u(x) := 1/\pi x$. The Hilbert transform of a function (or signal) $v(x)$ is given by

$$(Hv)(x) = \text{p.v.} \int_{-\infty}^{+\infty} u(x-y)v(y)dy,$$

provided this integral exists as Cauchy's principal value. This is precisely the Fourier convolution of v with the tempered distribution p.v. $u(x)$.

Put $r(x) := \dfrac{x}{2\sqrt{2\pi}(x^2 - h^2)}$, and $\breve{g}(x) := g(-x)$. By (2.1), we have

$$(u \underset{T_c}{\overset{\alpha}{*}} g)(x) = \text{p.v.}(r \underset{F}{*} g)(x) + \text{p.v.}(r \underset{F}{*} \breve{g})(x).$$

This means that the convolution (2.1) can be considered as a sum of the Fourier convolutions of r with the tempered distributions p.v. $g(x)$ and p.v. $\breve{g}(x)$. Similarly,

$$(u \underset{T_s}{\overset{\beta}{*}} g)(x) = \text{p.v.}(s \underset{F}{*} \breve{g})(x) - \text{p.v.}(s \underset{F}{*} g)(x), \quad \text{where} \quad s(x) := \frac{h}{2\sqrt{2\pi}(x^2 - h^2)}.$$

3. Application

3.1 Normed ring structures on $L^1(\mathbb{R}^d)$

This subsection deals with the construction of the normed ring structures on the space $L^1(\mathbb{R}^d)$ that could be used in the theories of Banach algebra (see [21]).

Definition 3.1. (see [19]) A vector space V with a ring structure and a vector norm is called a normed ring if $\|vw\| \le \|v\|\|w\|$, for all $v, w \in V$. If V has a multiplicative unit element e, it is also required that $\|e\| = 1$.

Let X denote the linear space $L^1(\mathbb{R}^d)$. For each of the convolutions in Section 2, the norm of $f \in X$ is chosen as

$$\|f\| = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |f(x)|dx.$$

THEOREM 3.1. *The space X, equipped with each of the convolution multiplications, becomes a normed ring having no unit.*

P r o o f. The proof is divided into two steps.

Step 1. X has a normed ring structure. It is clear that X, equipped with each of the convolution multiplications in Theorems 2.1 and 2.2, has the ring structure. We have to prove the multiplicative inequality. We now prove this assertion concerning the convolution (2.1), the proof being the same in the other cases.

Obviously,

$$\int_{\mathbb{R}^d} |f(x \pm u \pm h)|dx = \int_{\mathbb{R}^d} |f(x)|dx.$$

We then have

$$\begin{aligned}
\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |f \overset{\alpha}{\underset{T_c}{*}} g|(x)dx &\leq \frac{1}{4(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-u+h)||g(u)|dxdu \\
&+ \frac{1}{4(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-u-h)||g(u)|dxdu \\
&+ \frac{1}{4(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x+u+h)||g(u)|dxdu \\
&+ \frac{1}{4(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x+u-h)||g(u)|dxdu \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\int_{\mathbb{R}^d} |f(x)|dx \right) \left(\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |g(u)|du \right) = \|f\|\|g\|.
\end{aligned}$$

Hence, $\|f \overset{\alpha}{\underset{T_c}{*}} g\| \leq \|f\|.\|g\|$.

Step 2. X has no unit. For briefness of our proof, we use the common symbols: $*$ for the convolutions, and γ_0 for the weight functions α, β. Suppose that there exists an $e \in X$ such that $f = f * e = e * f$ for every $f \in X$. Choose $\delta(x) := e^{-\frac{1}{2}|x|^2} \in L^1(\mathbb{R}^d)$. Obviously, $(T_s\delta)(x) \equiv 0$. We then have $(F\delta)(x) = (\check{F}\delta)(x) = (T_c\delta)(x) = \delta(x)$ (see [21, Theorem 7.6]). By $\delta = \delta * e = e * \delta$ and the factorization identities of the convolutions, we have

$$\mathcal{T}_j(\delta) = \gamma_0(\mathcal{T}_k\delta)(\mathcal{T}_\ell e), \tag{3.1}$$

where $\mathcal{T}_j, \mathcal{T}_k, \mathcal{T}_\ell \in \{T_c, T_s\}$ (note that it may be $\mathcal{T}_j = \mathcal{T}_k = T_\ell = T_c$, etc.).

Proof for convolution (2.1). By (3.1), we have $\delta = \gamma_0\delta(T_c e)$. As $\delta(x) \neq 0$ for every $x \in \mathbb{R}^d$, $\gamma_0(x)(T_c e)(x) = 1$ for every $x \in \mathbb{R}^d$. Since $|\gamma_0(x)| \leq 1$, the last identity contradicts to the Riemann-Lebesgue lemma as: $\lim\limits_{x\to\infty}(T_c e)(x) = 0$ (see [21, Theorem 7.5]).

Proof for the convolutions (2.2), (2.3), (2.4). Using (3.1) and $(T_s\delta)(x) \equiv 0$, we have $(T_c\delta)(x) \equiv 0$. But, this fails.

Proof for the convolutions (2.5), (2.6), (2.8). Consider $\delta_0(x) = -2\frac{\partial\delta(x)}{\partial x_1} = 2x_1 e^{-\frac{1}{2}|x|^2}$. Obviously, $\delta_0(x) \in L^1(\mathbb{R}^d)$. Integrating by parts on variable y_1 we get

$$
\begin{aligned}
(T_c\delta_0)(x) &= \frac{-2}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\cos xy\left(\frac{\partial\delta(y)}{\partial y_1}\right)dy = \frac{-2x_1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\sin(xy)e^{-\frac{1}{2}|y|^2}dy = 0,\\
(T_s\delta_0(x) &= \frac{-2}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\sin xy\left(\frac{\partial\delta(y)}{\partial y_1}\right)dy = \frac{2x_1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\cos(xy)e^{-\frac{1}{2}|y|^2}dy\\
&= 2x_1(T_c\delta)(x) = 2x_1\delta(x).
\end{aligned}
$$

We now insert $\delta_0(x)$ into (3.1) and note that $\mathcal{T}_j = T_s, \mathcal{T}_k = T_c$ we obtain $x_1\delta(x) \equiv 0$, which fails.

Proof for convolution (2.7). Inserting $\delta_0(x)$ into (3.1), we get $2x_1\delta(x) = \gamma_0(x)2x_1\delta(x)(T_s e)(x)$. This implies $\gamma_0(x)(T_s e)(x) = 1$ for every $x_1 \neq 0$ which fails because $\lim\limits_{x_1,\ldots x_d\to\infty}\big(\gamma_0(x)(T_s e)(x)\big) = 0$.

Hence, X has no unit. The theorem is proved. ■

3.2 Integral equations of convolution type

The main aim of this section is to apply the convolutions in Section 2 for solving some integral equations of convolution type.

3.2.1 The Hartley transform

The multi-dimensional Hartley transform is defined as

$$(Hf)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\operatorname{cas}(xy)f(y)dy,$$

where $f(x)$ is a function (real or complex) defined on $\mathbb{R}^d$, and the integral kernel, known as the cosine-and-sine or *cas function*, is defined as $\operatorname{cas} xy = \cos xy + \sin xy$ (see [12]). The Hartley transform is a spectral transform closely related to the Fourier transform (see [1, 12]). The inversion theorem and some basic properties of the one-dimensional Hartley transform are well-known (see [1, 2, 3, 12, 18]). In this subsection we give a brief proof of the inversion theorem for the multi-dimensional Hartley transform and in Subsubsection 3.2.2 we show that it is useful for solving some integral equations.

Let $\mathcal{S}$ denote the set of all infinitely differentiable functions on $\mathbb{R}^d$ such that
$$\sup_{|\alpha|\le N}\sup_{x\in\mathbb{R}^d}(1+|x|^2)^N|(D_x^\alpha f)(x)|<\infty$$
for $N=0,1,2,\dots$ (see [21]). As F and F^{-1} are continuous linear maps of $\mathcal{S}$ into $\mathcal{S}$, H is also continuous (see [21, Theorem 7.7]).

Theorem 3.2. (inversion theorem, see [12]) *If $f\in L^1(\mathbb{R}^d)$, and if $Hf\in L^1(\mathbb{R}^d)$, then*
$$f_0(x):=\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}(Hf)(y)\,\mathrm{cas}(xy)dy=f(x)$$
for almost every $x\in\mathbb{R}^d$.

P r o o f. Let us first prove that if $g\in\mathcal{S}$, then
$$g(x)=\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}(Hg)(y)\,\mathrm{cas}(xy)dy. \tag{3.2}$$
Indeed, for any $\lambda>0$, put
$$B(0,\lambda):=\{y=(y_1,\dots,y_d)\in\mathbb{R}^d:|y_k|\le\lambda,\forall k=1,\dots,d\}$$
the d-dimensional box in $\mathbb{R}^d$. By induction on d, we can prove
$$\int_{B(0,\lambda)}[\cos y(x-t)+\sin y(x+t)]dy=\frac{2^d\sin\lambda(x_1-t_1)\dots\sin\lambda(x_d-t_d)}{(x_1-t_1)\dots(x_d-t_d)}.$$
Since $g\in\mathcal{S}$, Theorem 12 in [28] can be applied for this function. As the inner integral function $(Hg)(y)\,\mathrm{cas}\,xy$ on the right-side of (3.2) belongs to $\mathcal{S}$, the integral on the right side of (3.2) converges uniformly on $\mathbb{R}^d$ according to each of variables $x_1,\dots,x_d$. Therefore, we can use the Fubini's theorem, Theorem 12 in [28], and the above identity to calculate the integrals as follows
$$\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}(Hg)(y)\,\mathrm{cas}(xy)dy=\lim_{\lambda\to\infty}\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{B(0,\lambda)}\mathrm{cas}(xy)(Hg)(y)dy$$
$$=\lim_{\lambda\to\infty}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathrm{cas}(xy)\int_{B(0,\lambda)}\mathrm{cas}(yt)g(t)dtdy$$
$$=\lim_{\lambda\to\infty}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}g(t)\left(\int_{B(0,\lambda)}[\cos y(x-t)+\sin y(x+t)]dy\right)dt$$

$$= \frac{1}{(2\pi)^d} \lim_{\lambda\to\infty} \int_{\mathbb{R}^d} g(t) \frac{2^d \sin\lambda(x_1-t_1)\ldots\sin\lambda(x_d-t_d)}{(x_1-t_1)\ldots(x_d-t_d)} dt = g(x).$$

Thus, identity (3.2) is proved.

Let $g \in \mathcal{S}$ be given. Using Fubini's theorem, we get

$$\int_{\mathbb{R}^d} f(x)(Hg)(x)dx = \int_{\mathbb{R}^d} g(y)(Hf)(y)dy. \tag{3.3}$$

Inserting the inversion formula (3.2) into the right-side of (3.3) and using Fubini's theorem, we obtain

$$\int_{\mathbb{R}^d} f(x)(Hg)(x)dx = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} (Hg)(x) \operatorname{cas}(xy)dx \right) (Hf)(y)dy$$

$$= \int_{\mathbb{R}^d} (Hg)(x) \left(\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} (Hf)(y) \operatorname{cas}(xy)dy \right) dx = \int_{\mathbb{R}^d} f_0(x)(Hg)(x)dx.$$

By using (3.2), we can prove that transform H is a continuous, linear, one-to-one map of $\mathcal{S}$ onto $\mathcal{S}$, of period 2, whose inverse is continuous. Therefore, the functions Hg cover all of $\mathcal{S}$. We then have

$$\int_{\mathbb{R}^d} (f_0(x) - f(x))\Phi(x)dx = 0 \tag{3.4}$$

for every $\Phi \in \mathcal{S}$. Taking into account that $\mathcal{S}$ is dense in $L^1(\mathbb{R}^d)$, we conclude that $f_0(x) - f(x) = 0$ for almost every $x \in \mathbb{R}^d$. The theorem is proved. ■

Corollary 3.1. (uniqueness theorem) *If $f \in L^1(\mathbb{R}^d)$, and if $Hf = 0$ in $L^1(\mathbb{R}^d)$, then $f = 0$ in $L^1(\mathbb{R}^d)$.*

3.2.2 Integral equations with the mixed Toeplitz-Hankel kernel

Let $h_1, h_2 \in \mathbb{R}^d$ be given. Consider the integral equation of the form

$$\lambda\varphi(x) + \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} [k_1(x+y-h_1) + k_2(x-y-h_2)]\varphi(y)dy = p(x), \tag{3.5}$$

where $\lambda \in \mathbb{C}$ is predetermined, k_1, k_2, p are given, and $\varphi(x)$ is to be determined.

In what follows, given functions are assumed in $L^1(\mathbb{R}^d)$, and unknown function will be determined there. Therefore, the functional identity $f(x) = g(x)$ means that it is valid for almost every $x \in \mathbb{R}^d$. However, if both functions f, g are continuous, there should be emphasis that this identity must be true for every $x \in \mathbb{R}^d$.

Put $\gamma_1(x) := \cos xh_1$; $\gamma_2(x) := \sin xh_1$; $\gamma_3(x) := \cos xh_2$; $\gamma_4(x) := \sin xh_2$, and write:

$$\begin{aligned}&\mathbf{A}(x) := \lambda + \gamma_1(x)(T_c k_1)(x) - \gamma_2(x)(T_s k_1)(x) + \gamma_3(x)(T_c k_2)(x)\\&- \gamma_4(x)(T_s k_2)(x);\ \mathbf{B}(x) := \gamma_1(x)(T_s k_1)(x) + \gamma_2(x)(T_c k_1)(x)\\&- \gamma_3(x)(T_s k_2)(x) - \gamma_4(x)(T_c k_2)(x);\ \mathbf{C}(x) := \gamma_1(x)(T_s k_1)(x)+\\&\gamma_2(x)(T_c k_1)(x) + \gamma_3(x)(T_s k_2)(x) + \gamma_4(x)(T_c k_2)(x);\ \mathbf{D}(x) := \lambda-\\&\gamma_1(x)(T_c k_1)(x) + \gamma_2(x)(T_s k_1)(x) + \gamma_3(x)(T_c k_2)(x) - \gamma_4(x)(T_s k_2)(x);\\&D_{T_c}(x) := (T_c p)(x)\mathbf{D}(x) - (T_s p)(x)\mathbf{B}(x);\ D_{T_s}(x) := (T_s p)(x)\mathbf{A}(x)\\&- (T_c p)(x)\mathbf{C}(x);\ D_{T_c,T_s}(x) := \mathbf{A}(x)\mathbf{D}(x) - \mathbf{C}(x)\mathbf{B}(x).\end{aligned} \tag{3.6}$$

THEOREM 3.3. *Assume that* $D_{T_c,T_s}(x) \neq 0$ *for every* $x \in \mathbb{R}^d$, *and* $\frac{D_{T_c}}{D_{T_c,T_s}}$, $\frac{D_{T_s}}{D_{T_c,T_s}} \in L^1(\mathbb{R}^d)$. *Then equation* (3.5) *has solution in* $L^1(\mathbb{R}^d)$ *if and only if* $H\left(\frac{D_{T_c}+D_{T_s}}{D_{T_c,T_s}}\right) \in L^1(\mathbb{R}^d)$. *In this case the solution of the equation is given by* $\varphi(x) = H\left(\frac{D_{T_c}+D_{T_s}}{D_{T_c,T_s}}\right)(x)$.

P r o o f. Let us first prove the following lemma.

LEMMA 3.1. *Let* $f_1, f_2 \in L^1(\mathbb{R}^d)$. *Assume that* $f_1(x) = f_1(-x)$, *and* $f_2(x) = -f_2(-x)$, *for every* $x \in \mathbb{R}^d$. *Then* $H(f_1 + f_2)(x) = H(f_1 - f_2)(-x)$.

P r o o f. Obviously, $f_1 + f_2, f_1 - f_2 \in L^1(\mathbb{R}^d)$; $T_c f_2 = T_s f_1 = 0$; $(T_c f_1)(x) = (T_c f_1)(-x)$; and $(T_s f_2)(-x) = -(T_s f_2)(x)$. We then have $H(f_1+f_2)(x) = (T_c+T_s)(f_1+f_2)(x) = (T_c f_1)(x) + (T_s f_2)(x)$, and $H(f_1 - f_2)(-x) = (T_c+T_s)(f_1-f_2)(-x) = (T_c f_1)(-x) - (T_s f_2)(-x) = (T_c f_1)(x) + (T_s f_2)(x)$. The lemma is proved. ■

We now prove Theorem 3.3. Note that the shift h in the convolutions in Theorems 2.1, 2.2 is separate. From convolutions in Theorem 2.1 it follows that

$$\begin{aligned}\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(x+y-h_1)g(y)dy &= (f \overset{\gamma_1}{\underset{T_c}{*}} g)(x) + (f \overset{\gamma_1}{\underset{T_c,T_s,T_s}{*}} g)(x)\\&- (f \overset{\gamma_2}{\underset{T_c,T_s,T_c}{*}} g)(x) + (f \overset{\gamma_2}{\underset{T_c,T_c,T_s}{*}} g)(x),\\ \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(x-y-h_2)g(y)dy &= (f \overset{\gamma_3}{\underset{T_c}{*}} g)(x) - (f \overset{\gamma_3}{\underset{T_c,T_s,T_s}{*}} g)(x)\\&- (f \overset{\gamma_4}{\underset{T_c,T_s,T_c}{*}} g)(x) - (f \overset{\gamma_4}{\underset{T_c,T_c,T_s}{*}} g)(x).\end{aligned}$$

By the factorization identities of these convolutions, we have

$$T_c\left(\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(x+y-h_1)g(y)dy\right) = \gamma_1(x)\Big[(T_cf)(x)(T_cg)(x) + (T_sf)(x)(T_sg)(x)\Big] - \gamma_2(x)\Big[(T_sf)(x)(T_cg)(x) - (T_cf)(x)(T_sg)(x)\Big], \quad (3.7)$$

$$T_c\left(\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(x-y-h_2)g(y)dy\right) = \gamma_3(x)\Big[(T_cf)(x)(T_cg)(x) - (T_sf)(x)(T_sg)(x)\Big] - \gamma_4(x)\Big[(T_sf)(x)(T_cg)(x) + (T_cf)(x)(T_sg)(x)\Big]. \quad (3.8)$$

Similarly, by using the convolutions in Theorem 2.2, we have

$$T_s\left(\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(x+y-h_1)g(y)dy\right) = -\gamma_1(x)\Big[(T_cf)(x)(T_sg)(x) - (T_sf)(x)(T_cg)(x)\Big] + \gamma_2(x)\Big[(T_sf)(x)(T_sg)(x) + (T_cf)(x)(T_cg)(x)\Big], \quad (3.9)$$

$$T_s\left(\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(x-y-h_2)g(y)dy\right) = \gamma_3(x)\Big[(T_cf)(x)(T_sg)(x) + (T_sf)(x)(T_cg)(x)\Big] - \gamma_4(x)\Big[(T_sf)(x)(T_sg)(x) - (T_cf)(x)(T_cg)(x)\Big]. \quad (3.10)$$

Necessity. Suppose that equation (3.5) has a solution $\varphi \in L^1(\mathbb{R}^d)$. Applying T_c and T_s to both sides of the equation and using (3.7), (3.8), (3.9), (3.10), we obtain the system of two linear equations

$$\begin{cases} \big[\mathbf{A}(x)\big](T_c\varphi)(x) + \big[\mathbf{B}(x)\big](T_s\varphi)(x) = (T_cp)(x) \\ \big[\mathbf{C}(x)\big](T_c\varphi)(x) + \big[\mathbf{D}(x)\big](T_s\varphi)(x) = (T_sp)(x), \end{cases} \quad (3.11)$$

where $\mathbf{A}(x), \mathbf{B}(x), \mathbf{C}(x), \mathbf{D}(x)$ are defined as in (3.6), and $(T_c\varphi)(x), (T_s\varphi)(x)$ are the unknown functions. The determinants $D_{T_c,T_s}(x)$, $D_{T_c}(x), D_{T_s}(x)$ are determined as in (3.6). Since $D_{T_c,T_s}(x) \neq 0$ for every $x \in \mathbb{R}^d$, we find $(T_c\varphi)(x)$, $(T_s\varphi)(x)$. Unfortunately, the transforms T_c, T_s have no inverse transforms. We shall use the inverse formula of the Hartley transform to obtain the function $\varphi(x)$. By $D_{T_c,T_s}(x) \neq 0$ for every $x \in \mathbb{R}^d$, we get $(T_c\varphi)(x) = \frac{D_{T_c}(x)}{D_{T_c,T_s}(x)}$, $(T_s\varphi)(x) = \frac{D_{T_s}(x)}{D_{T_c,T_s}(x)}$. Hence, $(H\varphi)(x) = \frac{D_{T_c}(x)+D_{T_s}(x)}{D_{T_c,T_s}(x)}$. We now can use the inverse Hartley transform to get $\varphi(x) = H\left(\frac{D_{T_c}(x)+D_{T_s}}{D_{T_c,T_s}}\right)(x)$. Thus, $H\left(\frac{D_{T_c}+D_{T_s}}{D_{T_c,T_s}}\right) \in L^1(\mathbb{R}^d)$.

Sufficiency. Consider $\varphi(x) = H\left(\frac{D_{T_c}+D_{T_s}}{D_{T_c,T_s}}\right)(x)$. By (3.6), the functions $f_1(x) := \frac{D_{T_c}(x)}{D_{T_c,T_s}(x)}$, $f_2(x) := \frac{D_{T_s}(x)}{D_{T_c,T_s}(x)}$ satisfy the conditions of Lemma 3.1. We then have $\varphi(x) = H\left(\frac{D_{T_c}+D_{T_s}}{D_{T_c,T_s}}\right)(x) = H\left(\frac{D_{T_c}-D_{T_s}}{D_{T_c,T_s}}\right)(-x)$. Applying the inverse Hartley transform, we get

$$(H\varphi)(x) = \frac{D_{T_c}(x) + D_{T_s}(x)}{D_{T_c,T_s}(x)}, \quad (H\varphi)(-x) = \frac{D_{T_c}(x) - D_{T_s}(x)}{D_{T_c,T_s}(x)}.$$

As $(H\varphi)(x) = (T_c + T_s)\varphi)(x)$, and $(H\varphi)(-x) = (T_c - T_s)\varphi)(x)$, we find $(T_c\varphi)(x) = \frac{D_{T_c}(x)}{D_{T_c,T_s}(x)}$, $(T_s\varphi)(x) = \frac{D_{T_s}(x)}{D_{T_c,T_s}(x)}$. Hence, $(T_c\varphi)(x)$ and $(T_s\varphi)(x)$ fulfill (3.11). We thus have

$$\big[\mathbf{A}(x) + \mathbf{C}(x)\big](T_c\varphi)(x) + \big[\mathbf{B}(x) + \mathbf{D}(x)\big](T_s\varphi)(x) = (Hp)(x).$$

Equivalently,

$$H\Big(\lambda\varphi(x) + \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}[k_1(x+y-h_1) + k_2(x-y-h_2)]\varphi(y)dy\Big) = (Hp)(x).$$

By using the uniqueness theorem of the Hartley transform, φ fulfills equation (3.5) for almost every $x \in \mathbb{R}^d$ (see [12]). The theorem is proved. ∎

In the general theory of integral equations, the assumption that $D_{T_c,T_s}(x) \neq 0$ for every $x \in \mathbb{R}^d$ as in Theorem 3.3 is considered the normally solvable condition of the integral equation.

It is known that (3.5) is the Fredholm integral equation of first kind if $\lambda = 0$, and that of second kind if $\lambda \neq 0$. For the second kind, Proposition 3.1 below is the illustration of the conditions appearing in Theorem 3.3.

PROPOSITION 3.1. *Assume that $\lambda \neq 0$. Then*

(a) *$D_{T_c,T_s}(x) \neq 0$ for every x outside a ball with finite radius.*

(b) *If $D_{T_c,T_s}(x) \neq 0$ for every $x \in \mathbb{R}^d$, and if $T_cp, T_sp \in L^1(\mathbb{R}^d)$, then $\frac{D_{T_c}}{D_{T_c,T_s}}, \frac{D_{T_s}}{D_{T_c,T_s}} \in L^1(\mathbb{R}^d)$.*

P r o o f. (a) By the Riemann-Lebesgue lemma for the transforms T_c, T_s, the function $D_{T_c,T_s}(x)$ is continuous on $\mathbb{R}^d$, and $\lim\limits_{|x|\to\infty} D_{T_c,T_s}(x) = \lambda^2$. Now item (a) follows from $\lambda \neq 0$ and the continuity of $D_{T_c,T_s}(x)$.

(b) By the continuity of $D_{T_c,T_s}(x)$ and $\lim\limits_{|x|\to\infty} D_{T_c,T_s}(x) = \lambda^2 \neq 0$, there exist $R > 0$, $\varepsilon_1 > 0$ so that $\inf\limits_{|x|>R} |D_{T_c,T_s}(x)| > \varepsilon_1$. Since $D_{T_c,T_s}(x)$ is continuous, not vanished in the compact set: $S(0,R) = \{x \in \mathbb{R}^d : |x| \leq R\}$, there exists $\varepsilon_2 > 0$ so that $\inf\limits_{|x|\le R} |D_{T_c,T_s}(x)| > \varepsilon_2$. We then have $\sup\limits_{x\in\mathbb{R}^d} \frac{1}{|D_{T_c,T_s}(x)|} \le \max\{\frac{1}{\varepsilon_1}, \frac{1}{\varepsilon_2}\} < \infty$. This implies that the function $\frac{1}{|D_{T_c,T_s}(x)|}$ is continuous and bounded on $\mathbb{R}^d$. We now prove that if $T_c p, T_s p \in L^1(\mathbb{R}^d)$, then $D_{T_c}, D_{T_s} \in L^1(\mathbb{R}^d)$. Indeed, it is easily seen that the functions $\mathbf{A}(x), \mathbf{B}(x), \mathbf{C}(x), \mathbf{D}(x)$ are continuous and bounded on $\mathbb{R}^d$. This implies that $D_{T_c}, D_{T_s} \in L^1(\mathbb{R}^d)$. Thus, $\frac{D_{T_c}}{|D_{T_c,T_s}(x)|}, \frac{D_{T_s}}{|D_{T_c,T_s}(x)|} \in L^1(\mathbb{R}^d)$. The proposition is proved. ∎

Example 3.1. By calculating the right side of (3.6), we get

$$\begin{aligned} D_{T_c,T_s}(x) = \lambda^2 &+ 2\lambda[\gamma_3(x)(T_c k_2)(x) - \gamma_4(x)(T_s k_2)(x)] \\ &+ (T_c k_2)^2(x) + (T_s k_2)^2(x) - (T_c k_1)^2(x) - (T_s k_1)^2(x). \end{aligned}$$

If $k_1(x) = k_2(x) = e^{-\frac{|x|^2}{2}}$, then $D_{T_c,T_s}(x) = \lambda\big[\lambda + 2\gamma_3(x)e^{-\frac{|x|^2}{2}}\big]$. Therefore, $D_{T_c,T_s}(x) \neq 0$ for every $x \in \mathbb{R}^d$, provided $\lambda \in \mathbb{C} \setminus [-2, 2]$.

Comparison 3.1. (a) In constructing some generalized convolutions, the papers [17, 22, 23, 24, 25, 26, 27] solved their integral equations. Those papers provided the sufficient conditions for the solvability of the equations and obtained the implicit solutions of those equations via the Wiener-Lèvy theorem. By means of the normally solvable condition of an integral equation, the generalized convolutions in Section 2 work out the sufficient and necessary condition for the solvability of the equation (3.5) and its explicit solution via the Hartley transform.

(b) Observe that the convolutions in Section 2 do not contain any complex coefficient, and the Hartley transform of a real-valued function is real-valued rather than complex as is the case for the Fourier transform. Therefore, if the objects in integral equations are real-valued, then the use of the constructed convolutions and the Hartley transform brings about the remarkable advantage computationally (in the analysis of real signals) as it avoids the use of complex arithmetic (see [2, 3, 10, 20]).

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