VNU Journal of Science, Mathematics - Physics 23 (2007) 143-154

# On the martingale representation theorem and approximate hedging a contingent claim in the minimum mean square deviation criterion

Nguyen Van Huu[1,*], Vuong Quan Hoang[2]

[1] *Department of Mathematics, Mechanics, Informatics, College of Science, VNU*
*334 Nguyen Trai, Hanoi, Vietnam*
[2] *ULB Belgium*

Received 15 November 2006; received in revised form 12 September 2007

**Abstract.** In this work we consider the problem of the approximate hedging of a contingent claim in minimum mean square deviation criterion. A theorem on martingale representation in the case of discrete time and an application of obtained result for semi-continous market model are given.
*Keywords:* Hedging, contingent claim, risk neutral martingale measure, martingale representation.

## 1. Introduction

The activity of a stock market takes place usually in discrete time. Unfortunately such markets with discrete time are in general incomplete and so super-hedging a contingent claim requires usually an initial price two great, which is not acceptable in practice.

The purpose of this work is to propose a simple method for approximate hedging a contingent claim or an option in minimum mean square deviation criterion.

***Financial market model with discrete time:***

Without loss of generality let us consider a market model described by a sequence of random vectors $\{S_n,\ n = 0, 1, \ldots, N\}$, $S_n \in R^d$, which are discounted stock prices defined on the same probability space $\{\Omega,\ \Im,\ P\}$ with $\{F_n,\ n = 0, 1, \ldots, N\}$ being a sequence of increasing sigma-algebras of information available up to the time $n$, whereas "risk free " asset chosen as a numeraire $S_n^0 = 1$.

A $F_N$-measurable random variable $H$ is called a contingent claim (in the case of a standard call option $H = \max(S_n - K, 0)$, $K$ is strike price.

* Corresponding author. Tel.: 84-4-8542515.
E-mail: huunv@vnu.edu.vn

***Trading strategy:***

A sequence of random vectors of $d$-dimension $\gamma = (\gamma_n,\ n = 1, 2, \dots, N)$ with $\gamma_n = (\gamma_n^1, \gamma_n^2, \dots, \gamma_n^d)^T$ ($A^T$ denotes the transpose of matrix $A$ ), where $\gamma_n^j$ is the number of securities of type $j$ kept by the investor in the interval $[n-1,\ n)$ and $\gamma_n$ is $F_{n-1}$ -measurable (based on the information available up to the time $n-1$), then $\{\gamma_n\}$ is said to be predictable and is called *portfolio or trading strategy* .

***Assumptions:***

Suppose that the following conditions are satisfied:

i) $\Delta S_n = S_n - S_{n-1},\ H \in L_2(P),\ n = 0, 1, \dots, N.$

ii) Trading strategy $\gamma$ is self-financing, i.e. $S_{n-1}^T \gamma_{n-1} = S_{n-1}^T \gamma_n$ or equivalently $S_{n-1}^T \Delta\gamma_n = 0$ for all $n = 1, 2, \dots, N$.
Intuitively, this means that the portfolio is always rearranged in such a way its present value is preserved.

iii) The market is of *free arbitrage*, that means there is no trading strategy $\gamma$ such that $\gamma_1^T S_0 := \gamma_1.S_0 \leq 0,\ \gamma_N.S_N \geq 0,\ P\gamma_N.S_N > 0\} > 0.$

This means that with such trading strategy one need not an initial capital, but can get some profit and this occurs usually as the asset $\{Sn\}$ is not rationally priced.
Let us consider

$$G_N(\gamma) = \sum_{k=1}^{N} \gamma_k.\Delta S_k \text{ with } \gamma_k.\Delta S_k = \sum_{j=1}^{d} \gamma_k^j \Delta S_k^j.$$

This quantity is called the *gain* of the strategy $\gamma$ .

The problem is to find a constant $c$ and $\gamma = (\gamma_n,\ n = 1, 2, \dots, N)$ so that

$$E_P(H - c - G_N(\gamma))^2 \to \min. \tag{1}$$

Problem (1) have been investigated by several authors such as H.folmer, M.Schweiser, M.Schal, M.L.Nechaev with $d = 1$. However, the solution of problem (1) is very complicated and difficult for application if $\{S_n\}$ is not a $\{\mathbf{F}_n\}$-martingale under $P$, even for $d = 1$.

By the fundamental theorem of financial mathematics, since the market is of free arbitrage, there exists a probability measure $Q \sim P$ such that under $Q$ $\{S_n\}$ is an $\{\mathbf{F}_n\}$-martingale, i.e. $E_Q(S_n|F_n) = S_{n-1}$ and the measure $Q$ is called *risk neutral martingale probability measure* .

We try to find $c$ and $\gamma$ so that

$$E_Q(H - c - G_N(\gamma))^2 \to min \text{ over } \gamma. \tag{2}$$

**Definition 1.** $(\gamma_n^*) = (\gamma_n^*(c))$ *minimizing the expectation in (1.2) is called Q- optimal strategy in the minimum mean square deviation (MMSD) criterion corresponding to the initial capital c.*

The solution of this problem is very simple and the construction of the $Q$-optimal strategy is easy to implement in practice.

Notice that if $L_N = dQ/dP$ then

$$E_Q(H - c - G_N(\gamma))^2 = E_P[(H - c - G_N)^2 L_N]$$

can be considered as an weighted expectation under $P$ of $(H - c - G_N)^2$ with the weight $L_N$. This is similar to the pricing asset based on a risk neutral martingale measure $Q$.

In this work we give a solution of the problem (2) and a theorem on martingale representation in the case of discrete time.

It is worth to notice that the authors M.Schweiser, M.Schal, M.L.Nechaev considered only the problem (1) with $S_n$ of one-dimension and M.Schweiser need the additional assumptions that $\{S_n\}$ satisfies non-degeneracy condition in the sense that there exists a constant $\delta$ in $(0,1)$ such that

$$(E[\Delta S_n|F_{n-1}])^2 \leq \delta E[(\Delta S_n)^2|F_{n-1}] \qquad \text{P-a.s. for all } n = 1,2,\ldots,N.$$

and the trading strategies $\gamma_n$'s satisfy :

$$E[\gamma_n \Delta S_n]^2 < \infty,$$

while in this article $\{S_n\}$ is of $d$-dimension and we need not the preceding assumptions.

The organization of this article is as follows:

The solution of the problem (2) is fulfilled in paragraph 2.(Theorem 1) and a theorem on the representation of a martingale in terms of the differences $\Delta S_n$ (Theorem 2) will be also given (the representation is similar to the one of a martingale adapted to a Wiener filter in the case of continuous time).

Some examples are given in paragraph 3.

The semi-continuous model, a type of discretization of diffusion model, is investigated in paragraph 4.

## 2. Finding the optimal portfolio

**Notation.** Let $Q$ be a probability measure such that $Q$ is equivalent to $P$ and under $Q$ $\{S_n,\ n = 1,2,\ldots,N\}$ is an integrable square martingale and let us denote $E_n(X) = E_Q(X|F_n)$, $H_N = H$, $H_n = E_Q(H|F_n) = E_n(H)$; $\text{Var}_{n-1}(X) = [\text{Cov}_{n-1}(X_i, X_j)]$ denotes the conditional variance matrix of random vector $X$ when $F_{n-1}$ is given, $\Gamma$ is the family of all predictable strategies $\gamma$.

**Theorem 1.** *If $\{S_n\}$ is an $\{F_n\}$-martingale under $Q$ then*

$$E_Q(H - H_0 - G_N(\gamma^*))^2 = \min\{E_Q(H - c - G_N(\gamma))^2 : \gamma \in \Gamma\}, \tag{3}$$

*where $\gamma_n^*$ is a solution of the following equation system:*

$$[Var_{n-1}(\Delta S_n)]\gamma_n^* = E_{n-1}((\Delta H_n \Delta S_n) \qquad P\text{- a.s.}, \tag{4}$$

*Proof.* At first let us notice that the right side of (3) is finite. In fact, with $\gamma_n = 1$ for all $n$, we have

$$E_Q(H - c - G_N(\gamma))^2 = E_Q\left(H - c - \sum_{n=1}^{N}\sum_{j=1}^{d} \Delta S_n^j\right)^2 < \infty.$$

Furthermore, we shall prove that $\gamma^* \Delta S_n$ is integrable square under $Q$.

Recall that (see [Appendix A]) if $Y, X_1, X_2, \ldots, X_d$ are $d+1$ integrable square random variables with $E(Y) = E(X_1) = \cdots = E(X_d) = 0$ and if $\widehat{Y} = b_1X_1 + b_2X_2 + \cdots + b_dX_d$ is the optimal linear predictor of $Y$ on the basis of $X_1, X_2, \ldots, X_d$ then the vector $b = (b_1, b_2, \ldots, b_d)^T$ is the solution of the following equations system :

$$\text{Var}(X)b = E(YX), \tag{5}$$

and as Var($X$) is non-degenerated b is defined by

$$b = [\mathrm{Var}(X)]^{-1}E(YX), \tag{6}$$

and in all cases

$$b^T E(YX) \le E(Y^2), \tag{7}$$

where $X = (X_1, X_2, \dots, X_k)^T$.

Furthermore,

$$Y - \widehat{Y} \perp X_i, \text{ i.e. } E[X_i(Y - \widehat{Y})] = 0,\ i = 1, \dots, k. \tag{8}$$

Applying the above results to the problem of conditional linear prediction of $\Delta H_n$ on the basis of $\Delta S_n^1, \Delta S_n^2, \dots, \Delta S_n^d$ as $F_n$ is given we obtain from (5) the formula (4) defining the regression coefficient vector $\gamma^*$. On the other hand we have from (5) and (7):

$$\begin{aligned} E(\gamma_n^{*T}\Delta S_n)^2 &= EE_{n-1}(\gamma_n^{*T}\Delta S_n \Delta S_n^T \gamma_n^{*T}) = E(\gamma_n^{*T}\mathrm{Var}_{n-1}(\Delta S_n)\gamma_n) \\ &= E(\gamma_n^* E_{n-1}(\Delta H_n \Delta S_n)) \le E(\Delta H_n)^2 < \infty. \end{aligned}$$

With the above remarks we can consider only, with no loss of generality, trading strategies $\gamma_n$ such that

$$E_{n-1}(\gamma_n \Delta S_n)^2 < \infty.$$

We have:

$$H_N = H_0 + \Delta H_1 + \cdots + \Delta H_N$$

and

$$E_{n-1}(\Delta H_n - \gamma_n^T \Delta S_n)^2 = E_{n-1}(\Delta H_n)^2 - 2\gamma_n^T E_{n-1}((\Delta H_n \Delta S_n) + \gamma_n^T E_{n-1}(\Delta S_n \Delta S_n^T)\gamma_n.$$

This expression takes the minimum value when $\gamma_n = \gamma_n^*$.

Furthermore, since $\{H_n - c - G_n(\gamma)\}$ is an $\{F_n\}$- integrable square martingale under $Q$,

$$\begin{aligned} E_Q(H_N - c - G_N(\gamma))^2 &= E_Q\left[H_0 - c - \sum_{n=1}^{N}(\Delta H_n - \gamma_n \Delta S_n)\right]^2 \\ &= (H_0 - c)^2 + E_Q\left[\sum_{n=1}^{N}(\Delta H_n - \gamma_n \Delta S_n)\right]^2 \\ &= (H_0 - c)^2 + \sum_{n=1}^{N} E_Q(\Delta H_n - \gamma_n \Delta S_n)^2 \text{ (for } \Delta H_n - \gamma_n \Delta S_n \text{ being a martingale difference)} \\ &= (H_0 - c)^2 + E_Q \sum_{n=1}^{N} E_{n-1}(\Delta H_n - \gamma_n \Delta S_n)^2 \\ &\ge (H_0 - c)^2 + E_Q \sum_{n=1}^{N} E_{n-1}(\Delta H_n - \gamma_n^* \Delta S_n)^2 \end{aligned}$$

$$= (H_0 - c)^2 + E_Q \sum_{n=1}^{N} (\Delta H_n - \gamma_n^* \Delta S_n)^2$$

$$= (H_0 - c)^2 + E_Q \left[ \sum_{n=1}^{N} (\Delta H_n - \gamma_n^* \Delta S_n) \right]^2$$

$$\geq E_Q(H_N - H_0 - G_n(\gamma^*))^2.$$

So $E_Q(H_N - c - G_N(\gamma))^2 \geq E_Q(H_N - H_0 - G_n(\gamma^*))^2$ and the inequality becomes the equality if $c = H_0$ and $\gamma = \gamma^*$.

## 3. Martingale representation theorem

**Theorem 2.** *Let* $\{H_n,\ n = 0, 1, 2, \ldots\}$, $\{S_n,\ n = 0, 1, 2, \ldots\}$ *be arbitrary integrable square random variables defined on the same probability space* $\{\Omega, \Im, \mathbf{P}\}$, $F_n^S = \sigma(S_0, \ldots, S_n)$. *Denoting by* $\Pi(S, P)$ *the set of probability measures* $Q$ *such that* $Q \sim P$ *and that* $\{S_n\}$ *is* $\{F_n^S\}$ *integrable square martingale under* $Q$, *then if* $F = \bigvee_{n=0}^{\infty} F_n^S$, $H_n,\ S_n \in L_2(Q)$ *and if* $\{H_n\}$ *is also a martingale under* $Q$ *we have:*

$$1.\ H_n = H_0 + \sum_{k=1}^{n} \gamma_k^T \Delta S_k + C_n \qquad a.s., \tag{9}$$

*where* $\{C_n\}$ *is a* $\{F_n^S\}-Q$*-martingale orthogonal to the martingale* $\{S_n\}$, *i.e.* $E_{n-1}((\Delta C_n \Delta S_n) = 0$, *for all* $n = 0, 1, 2, \ldots$, *whereas* $\{\gamma_n\}$ *is* $\{F_{n-1}^S\}$*- predictable.*

$$2.\ H_n = H_0 + \sum_{k=1}^{n} \gamma_k^T \Delta S_k := H_0 + G_n(\gamma) \qquad P\text{-a..s.} \tag{10}$$

*for all* $n$ *finite iff the set* $\Pi(S, P)$ *consists of only one element.*

*Proof.* According to the proof of Theorem 1, Putting

$$\Delta C_k = \Delta H_k - \gamma_k^{*T} \Delta S_k,\ C_n = \sum_{k=1}^{n} \Delta C_k,\ C_0 = 0, \tag{11}$$

then $\Delta C_k \perp \Delta S_k$, by (8).

Taking summation of (11) we obtain (9).

The conclusion 2 follows from the fundamental theorem of financial mathematics.

**Remark 3.1.** By the fundamental theorem of financial mathematics a security market has no arbitrage opportunity and is complete iff $\Pi(S, P)$ consists of the only element and in this case we have (10) with $\gamma$ defined by (4). Furthermore, in this case the conditional probability distribution of $S_n$ given $F_{n-1}^S$ concentrates at most $d + 1$ points of $R^d$ (see [2], [3]), in particular for $d = 1$, with exception of binomial or generalized binomial market models (see [2], [7]), other models are incomplete.

**Remark 3.2.** We can choose the risk neutral martingale probability measure $Q$ so that $Q$ has minimum entropy in $\Pi(S, P)$ as in [2] or $Q$ is near $P$ as much as possible.

**Example 1.** Let us consider a stock with the discounted price $S_0$ at $t = 0$, $S_1$ at $t = 1$, where

$$S_1 = \begin{cases} 4S_0/3 & \text{with prob. } p_1, \\ S_0 & \text{with prob. } p_2, \quad p_1, p_2, p_3 > 0, \quad p_1 + p_2 + p_3 = 1 \\ 5S_0/6 & \text{with prob. } p_3. \end{cases}$$

Suppose that there is an option on the above stock with the maturity at $t = 1$ and with strike price $K = S_0$. We shall show that there are several probability measures $Q \sim P$ such that $\{S_0, S_1\}$ is, under $Q$, a martingale or equivalently $E_Q(\Delta S_1) = 0$.

In fact, suppose that $Q$ is a probability measure such that under $Q$ $S_1$ takes the values $4S_0/3$, $S_0$, $2S_0/3$ with positive probability $q_1$, $q_2$, $q_3$ respectively. Then $E_Q(\Delta S_1) = 0 \Leftrightarrow S_0(q_1/3 - q_3/6) = 0 \Leftrightarrow 2q_1 = q_3$, so $Q$ is defined by $(q_1,\ 1 - 3q_1,\ 2q_1)$, $0 < q_1 < 1/3$.

In the above market, the payoff of the option is

$$H = (S_1 - K)_+ = (\Delta S_1)_+ = \max(\Delta S_1, 0).$$

It is easy to get an $Q$-optimal portfolio

$$\gamma^* = E_Q[H\Delta S_1]/E_Q(\Delta S_1)^2 = 2/3,\ E_Q(H) = q_1 S_0/3,$$

$$E_Q[H - E_Q(H) - \gamma^*\Delta S_1]^2 = q_1 S_0^2(1 - 3q_1)/9 \to 0 \text{ as } q_1 \to 1/3.$$

However we can not choose $q_1 = 1/3$, because $q = (1/3,\ 0,\ 2/3)$ is not equivalent to $P$. It is better to choose $q1 \cong 1/3$ and $0 < q_1 < 1/3$.

**Example 2.** Let us consider a market with one risky asset defined by :

$$S_n = S_0 \prod_{i=1}^{n} Z_i, \text{ or } S_n = S_{n-1}Z_n,\ n = 1, 2, \ldots, N,$$

where $Z_1, Z_2, \ldots, Z_N$ are the sequence of i.i.d. random variables taking the values in the set $\Omega = \{d_1, d_2, \ldots, d_M)$ and $P(Z_i = d_k) = p_k > 0$, $k = 1, 2, \ldots, M$. It is obvious that a probability measure $Q$ is equivalent to $P$ and under $Q$ $\{S_n\}$ is a martingale if and only if $Q\{Z_i = d_k) = q_k > 0$, $k = 1, 2, \ldots, M$ and $E_Q(Z_i) = 1$ , i.e.

$$q_1 d_1 + q_2 d_2 + \cdots + q_M d_M = 1.$$

Let us recall the integral Hellinger of two measure $Q$ and $P$ defined on some measurable space $\{\Omega^*, F\}$:

$$H(P, Q) = \int_{\Omega^*} (dP.dQ)^{1/2}.$$

In our case we have

$$H(P, Q) = \sum \{P(Z_1 = d_{i1},\ Z_2 = d_{i2}, \ldots, Z_N = d_{iN})^* Q(Z_1 = d_{i1},\ Z_2 = d_{i2}, \ldots, Z_N = d_{iN})^{1/2}$$
$$= \sum \{p_{i1}q_{i1}\ p_{i2}q_{i2} \cdots p_{iN}q_{iN}\}^{1/2}$$

where the summation is extended over all $d_{i1}, d_{i2}, \ldots, d_{iN}$ in $\Omega$ or over all $i_1, i_2, \ldots, i_N$ in $\{1, 2, \ldots, M\}$. Therefore

$$H(P, Q) = \left\{ \sum_{i=1}^{M} (p_i q_i)^{1/2} \right\}^N .$$

We can define a distance between $P$ and $Q$ by

$$||Q - P||^2 = 2(1 - H(P, Q)).$$

Then we want to choose $Q^*$ in $\Pi(S, P)$ so that $||Q^* - P|| = \inf\{||Q - P|| : Q \in \Pi(S, P)\}$ by solving the following programming problem:

$$\sum_{i=1}^{M} p_i^{1/2} q_i^{1/2} \to \max$$

with the constraints :

i) $q_1 d_1 + q_2 d_2 + \cdots + q_M d_M = 1$.
ii) $q_1 + q_2 + \cdots + q_M = 1$.
iii) $q_1, q_2, \ldots, q_M > 0$.

Giving $p_1, p_2, \ldots, p_M$ we can obtain a numerical solution of the above programming problem. It is possible that the above problem has not a solution. However, we can replace the condition (3) by the condition

iii') $q_1, q_2, \ldots, q_d \geq 0$,

then the problem has always the solution $q^* = (q_1^*, q_2^*, \ldots, q_M^*)$ and we can choose the probabilities $q_1, q_2, \ldots, q_M > 0$ are sufficiently near to $q_1^*, q_2^*, \ldots, q_M^*$ .

## 4. Semi-continuous market model (discrete in time continuous in state)

Let us consider a financial market with two assets:

+ Free risk asset $\{B_n, n = 0, 1, \ldots, N\}$ with dynamics

$$B_n = \exp\left(\sum_{k=1}^{n} r_k\right), \ 0 < r_n < 1. \tag{12}$$

+ Risky asset $\{S_n, n = 0, 1, \ldots, N\}$ with dynamics

$$S_n = S_0 \exp\left(\sum_{k=1}^{n} [\mu(S_{k-1}) + \sigma(S_{k-1}) g_k]\right), \tag{13}$$

where $\{g_n, n = 0, 1, \ldots, N\}$ is a sequence of i.i.d. normal random variable $N(0, 1)$. It follows from (13) that

$$S_n = S_{n-1} \exp(\mu(S_{n-1}) + \sigma(S_{n-1}) g_n), \tag{14}$$

where $S_0$ is given and $\mu(S_{n-1}) := a(S_{n-1}) - \sigma^2(S_{n-1})/2$, with $a(x)$, $\sigma(x)$ being some functions defined on $[0, \infty)$ .

The discounted price of risky asset $\tilde{S}_n = S_n/B_n$ is equal to

$$\tilde{S}_n = S_0 \exp\left(\sum_{k=1}^{n} [\mu(S_{k-1}) - r_k + \sigma(S_{k-1}) g_k]\right). \tag{15}$$

We try to find a martingale measure $Q$ for this model.

It is easy to see that $E_P(\exp(\lambda g_k)) = exp(\lambda^2/2)$, for $g_k \sim N(0, 1)$, hence

$$E \exp\left(\sum_{k=1}^{n} [\beta_k(S_{k-1}) g_k - \beta_k(S_{k-1})^2/2]\right) = 1 \tag{16}$$

for all random variable $\beta_k(S_{k-1})$ .

Therefore, putting

$$L_n = \exp\left(\sum_{k=1}^{n}[\beta_k(S_{k-1})g_k - \beta_k(S_{k-1})^2/2]\right),\ n = 1,\dots,N \tag{17}$$

and if $Q$ is a measure such that $dQ = L_N dP$ then $Q$ is also a probability measure. Furthermore,

$$\frac{\tilde{S_n}}{\tilde{S_{n-1}}} = \exp(\mu(S_{n-1}) - r_n + \sigma(S_{n-1})g_n). \tag{18}$$

Denoting by $E^0$, $E$ expectation operations corresponding to $P$, $Q$, $E_n(.) = E[(.)|F_n^S]$ and choosing

$$\beta_n = -\frac{(a(S_{n-1}) - r_n)}{\sigma(S_{n-1})} \tag{19}$$

then it is easy to see that

$$E_{n-1}[\tilde{S_n}/\tilde{S_{n-1}}] = E^0[L_n\tilde{S_n}/\tilde{S_{n-1}}|F_n^S]/L_{n-1} = 1$$

which implies that $\{\tilde{S_n}\}$ is a martingale under $Q$.

Furthermore, under $Q$, $S_n$ can be represented in the form

$$S_n = S_{n-1}\exp((\mu^*(S_{n-1}) + \sigma(S_{n-1})g_n^*). \tag{20}$$

Where $\mu^*(S_{n-1}) = r_n - \sigma^2(S_{n-1})/2$, $g_n^* = -\beta_n + g_n$ is Gaussian $N(0,1)$. It is not easy to show the structure of $\Pi(S,P)$ for this model.

We can choose a such probability measure $E$ or the weight function $L_N$ to find a $Q$- optimal portfolio.

**Remark 4.3.** The model (12), (13) is a type of discretization of the following diffusion model:

Let us consider a financial market with continuous time consisting of two assets:

+Free risk asset:

$$B_t = \exp\left(\int_0^t r(u)du\right). \tag{21}$$

+Risky asset: $dS_t = S_t[a(S_t)dt + \sigma(S_t)dWt]$, $S_0$ is given, where $a(.),\ \sigma(.) : (0,\infty) \to R$ such that $xa(x)$, $x\sigma(x)$ are Lipschitz. It is obvious that

$$S_t = \exp\left\{\int_0^t [a(S_u) - \sigma^2(S_u)/2]du + \int_0^t \sigma(S_u)dW_u\right\},\ 0 \le t \le T. \tag{22}$$

Putting

$$\mu(S) = a(S) - \sigma^2(S)/2, \tag{23}$$

and dividing $[0,T]$ into $N$ intervals by the equidistant dividing points $0, \Delta, 2\Delta, \dots, N\Delta$ with $N = T/\Delta$ sufficiently great, it follows from (21), (22) that

$$\begin{aligned} S_{n\Delta} &= S_{(n-1)\Delta}\exp\left\{\int_{(n-1)\Delta}^{n\Delta}\mu(S_u)du + \int_{(n-1)\Delta}^{n\Delta}\sigma(S_u)dW_u\right\} \\ &\cong S_{(n-1)\Delta}\exp\{\mu(S_{(n-1)\Delta})\Delta + (S_{(n-1)\Delta})[W_{n\Delta} - W_{(n-1)\Delta}]\} \\ &\cong S_{(n-1)\Delta}\exp\{\mu(S_{(n-1)\Delta})\Delta + \sigma(S_{(n-1)\Delta})\Delta^{1/2}g_n\} \end{aligned}$$

with $g_n = [W_{n\Delta} - W_{(n-1)\Delta}]/\Delta^{1/2}$, $n = 1, \ldots, N$, being a sequence of the i.i.d. normal random variables of the law $N(0,1)$, so we obtain the model :

$$S^*_{n\Delta} = S^*_{(n-1)\Delta} \exp\{\mu(S^*_{(n-1)\Delta})\Delta + \sigma(S^*_{(n-1)\Delta})\Delta^{1/2} g_n\}. \tag{24}$$

Similarly we have

$$B^*_{n\Delta} \cong B^*_{(n-1)\Delta} \exp(r_{(n-1)\Delta}\Delta). \tag{25}$$

According to (21), the discounted price of the stock $S_t$ is

$$\tilde{S}_t = \frac{S_t}{B_t} = S_0 \exp\left\{\int_0^t [\mu(S_u) - r_u] d_u + \int_0^t \sigma(S_u) dW_u\right\}. \tag{26}$$

By Theorem Girsanov, the unique probability measure $Q$ under which $\{\tilde{S}_t,\ F_t^S,\ Q\}$ is a martingale is defined by

$$(dQ/dP)|F_T^S = \exp\left(\int_0^T \beta_u dW_u \ - \frac{1}{2}\int_0^T \beta_u^2 d_u\right) := L_T(\omega), \tag{27}$$

where

$$\beta_s = -\frac{((a(S_s) - r_s)}{\sigma(S_s)},$$

and $(dQ/dP)|F_T^S$ denotes the Radon-Nikodym derivative of $Q$ w.r.t. $P$ limited on $F_T^S$. Furthermore, under $Q$

$$W_t^* = W_t + \int_0^t \beta_u du$$

is a Wiener process. It is obvious that LT can be approximated by

$$L_N := \exp\left(\sum_{k=1}^N \beta_k \Delta^{1/2} g_k - \Delta\beta_k^2/2\right) \tag{28}$$

where

$$\beta_n = -\frac{[a(S_{(n-1)\Delta}) - r_{n\Delta}]}{\sigma(S_{(n-1)\Delta})} \tag{29}$$

Therefore the weight function (25) is approximate to Radon-Nikodym derivative of the risk unique neutral martingale measure $Q$ w.r.t. $P$ and $Q$ is used to price derivatives of the market.

Remark 4.4. In the market model Black- Scholes we have $L_N = L_T$. We want to show now that for the weight function (28)

$$E_Q(H - H_0 - G_N(\gamma^*))^2 \to 0 \text{ as } N \to \infty \text{ or } \Delta \to 0.$$

where $\gamma^*$ is $Q$-optimal trading strategy.

**Proposition.** *Suppose that $H = H(S_T)$ is a integrable square discounted contingent claim. Then*

$$E_Q(H - H_0 - G_N(\gamma^*))^2 \to 0 \text{ as } N \to \infty \text{ or } \Delta \to 0, \tag{30}$$

*provided $a$, $r$ and $\sigma$ are constant ( in this case the model* (21), (22) *is the model Black-Scholes ).*

*Proof.* It is well known (see[4], [5]) that for the model of complete market (21), (22) there exists a trading strategy $\varphi = (\varphi_t = \varphi(t, S(t)),\ 0 = t = T)$, hedging $H$, where $\varphi : [0,T] \times (0,\infty) \to R$ is continuously derivable in $t$ and $S$, such that

$$H(S_T) = H_0 + \int_0^T \varphi_t d\tilde{S}(t) \qquad \text{a.s.}$$

On the other hand we have

$$E_{Q_N}\left(H - H_0 - \sum_{k=1}^{N}\gamma^{*}_{(k-1)\Delta}\Delta\tilde{S}_{n\Delta}\right)^2$$

$$\leq E_{Q_N}\left(H - H_0 - \sum_{k=1}^{N}\varphi_{(k-1)\Delta}\Delta\tilde{S}_{n\Delta}\right)^2$$

$$= E_Q\left(\int_0^T \varphi_t d\tilde{S}(t) - \sum_{k=1}^{N}\varphi_{(n-1)\Delta}\Delta\tilde{S}_{(n-1)\Delta}\right)^2 L_N/L_T$$

$$= E_Q\left(\int_0^T \varphi_t d\tilde{S}(t) - \sum_{k=1}^{N}\phi_{(k-1)\Delta}\Delta\tilde{S}_{(n-1)\Delta}\right)^2 \to 0 \text{ as } \Delta \to 0.$$

(since $L_N = L_T$ and by the definition of the stochastic integral Ito as $a$ and $\sigma$ are constant ) .

**Appendix A**

Let $Y, X_1, X_2, \ldots, X_d$ be integrable square random variables defined on the same probability space $\{\Omega, F, P\}$ such that $EX_1 = \cdots = EX_d = EY = 0$ .

We try to find a coefficient vector $b = (b_1, \ldots, b_d)^T$ so that

$$E(Y - b_1X_1 - \cdots - b_dX_d)^2 = E(Y - b^TX)^2 = \min_{a\in R^d}(Y - a^TX)^2. \quad \text{(A1)}$$

Let us denote $EX = (EX_1, \ldots, EX_d)^T$, $\text{Var}(X) = [\text{Cov}(X_i, X_j),\ i, j = 1, 2, \ldots, d] = EXX^T$.

**Proposition.** nghieng The vector $b$ minimizing $E(Y - a^TX)^2$ is a solution of the following equation system :

$$\text{Var}(X)b = E(XY). \quad \text{(A2)}$$

Putting $U = Y - b^TX = Y - \hat{Y}$, with $\hat{Y} = b^TX$, then

$$E(U^2) = EY^2 - b^TE(XY) \geq 0. \quad \text{(A3)}$$

$$E(UX_i) = 0 \ \text{ for all } i = 1, \ldots, d. \quad \text{(A4)}$$

$$EY^2 = EU^2 + E\hat{Y}^2. \quad \text{(A5)}$$

$$\rho = \frac{EY\hat{Y}}{(EY^2E\hat{Y}^2)^{1/2}} = \left(\frac{E\hat{Y}^2}{EY^2}\right)^{1/2} \quad \text{(A6)}$$

($\rho$ is called multiple correlation coefficient of $Y$ relative to $X$).

*Proof.* Suppose at first that Var($X$) is a positively definite matrix. For each $a \in R^d$ We have

$$F(a) = E(Y - a^TX)^2 = EY^2 - 2a^TE(XY) + a^TEXX^Ta \quad \text{(A7)}$$

$$\nabla F(a) = -2E(XY) + 2\text{Var}(X)a.$$

$$\left[\frac{\partial F(a)}{\partial a_i \partial a_j},\ i, j = 1, 2, \ldots, d\right] = 2\text{Var(X)}.$$

It is obvious that the vector b minimizing $F(a)$ is the unique solution of the following equation:

$$\nabla F(a) = 0 \text{ or (A2)}$$

and in this case (A2) has the unique solution :

$$b = [\mathrm{Var}(X)]^{-1}E(XY).$$

We assume now that $1 \leq \mathrm{Rank}(\mathrm{Var}(X)) = r < d$.

We denote by $e_1, e_2, \ldots, e_d$ the ortho-normal eigenvectors w.r.t. the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_d$ of $\mathrm{Var}(X)$ , where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0 = \lambda_{r+1} = \cdots = \lambda_d$ and $P$ is a orthogonal matrix with the columns being the eigenvectors $e_1, e_2, \ldots, e_d$, then we obtain :

$$\mathrm{Var}(X) = P\Lambda P^T, \text{ with } \Lambda = \mathrm{Diag}(\lambda_1, \lambda_2, \ldots, \lambda_d).$$

Putting

$$Z = P^T X = [e_1^T X, e_2^T X, \ldots, e_d^T X]^T,$$

$Z$ is the principle component vector of $X$, we have

$$\mathrm{Var}(Z) = P^T \mathrm{Var}(X) P = \Lambda = \mathrm{Diag}(\lambda_1, \lambda_2, \ldots, \lambda_r, 0, \ldots, 0).$$

Therefore

$$EZ_{r+1}^2 = \cdots = EZ_d^2 = 0, \text{ so } Z_{r+1} = \cdots = Z_d = 0 \text{ P- a.s.}$$

Then

$$\begin{aligned} F(a) = E(Y - a^T X)^2 &= E(Y - (a^T P)Z)^2 \\ &= E(Y - a_1^* Z_1 - \cdots - a_d^* Z_d)^2 \\ &= E(Y - a_1^* Z_1 - \cdots - a_r^* Z_r)^2. \end{aligned}$$

where

$$a^{*T} = (a_1^*, \ldots, a_d^*) = a^T P, \quad \mathrm{Var}(Z_1, \ldots, Z_r) = \mathrm{Diag}(\lambda_1, \lambda_2, \ldots, \lambda_r) > 0.$$

According to the above result $(b_1^*, \ldots, b_r^*)^T$ minimizing $E(Y - a_1^* Z_1 - \cdots - a_r^* Z_r)^2$ is the solution of

$$\begin{pmatrix} \lambda_1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & \lambda_r \end{pmatrix} \begin{pmatrix} b_1^* \\ \ldots \\ b_r^* \end{pmatrix} = \begin{pmatrix} EZ_1Y \\ \ldots \\ EX_rY \end{pmatrix} \tag{A8}$$

or

$$\begin{pmatrix} \lambda_1 & \ldots & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \lambda_r & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 & \ldots & 0 \end{pmatrix} \begin{pmatrix} b_1^* \\ \ldots \\ b_r^* \\ b_{r+1}^* \\ \ldots \\ b_d^* \end{pmatrix} = \begin{pmatrix} EZ_1Y \\ \ldots \\ EZ_rY \\ 0 \\ \ldots \\ 0 \end{pmatrix} = \begin{pmatrix} EZ_1Y \\ \ldots \\ EZ_rY \\ EZ_{r+1}Y \\ \ldots \\ EZ_dY \end{pmatrix} \tag{A9}$$

with $b_{r+1}^*, \ldots, b_d^*$ arbitrary .

Let $b = (b_1, \ldots, b_d)^T$ be the solution of $b^T P = b^{*T}$, hence $b = Pb^*$ with $b^*$ being a solution of (A9). Then it is follows from (A9) that

$$\mathrm{Var}(Z)P^T b = E(ZY) = P^T E(XY)$$

or

$$P^T \mathrm{Var}(X) P P^T b = P^T E(XY) \text{ ( since } \mathrm{Var}(Z) = P^T \mathrm{Var}(X) P)$$

or

$$\mathrm{Var}(X) b = E(XY)$$

which is (A2). Thus we have proved that (A2) has always a solution ,which solves the problem (A1). By (A7) , we have

$$\begin{aligned} F(b) &= \min_a E(Y - a^T X)^2 \\ &= EY^2 - 2b^T E(XY) + b^T \mathrm{Var}(X) b \\ &= EY^2 - 2b^T E(XY) + b^T E(XY) \\ &= EY^2 - b^T E(XY) \geq 0. \end{aligned}$$

On the other hand

$$EUX_i = E(X_i Y) - E(X_i b^T X) = 0, \quad \text{(A10)}$$

since $b$ is a solution of (A2) and (A10) is the ith equation of the system (A2).
It follows from (A10) that

$$E(U\hat{Y}) = 0 \text{ and } EY^2 = E(U + \hat{Y})^2 = EU^2 + E\hat{Y}^2 + 2E(U\hat{Y}) = EU^2 + E\hat{Y}^2.$$

**Remark.** We can use Hilbert space method to prove the above proposition. In fact, let $H$ be the set of all random variables $\xi$'s such that $E\xi = 0$, $E\xi^2 < \infty$, then $H$ becomes a Hilbert space with the scalar product $(\xi, \zeta) = E\xi\zeta$, and with the norm $||\xi|| = (E\xi^2)^{1/2}$ . Suppose that $X_1, X_2, \ldots, X_d$, $Y \in H$, $L$ is the linear manifold generated by $X_1, X_2, \ldots, X_d$ . We want to find a $\hat{Y} \in H$ so that $||Y - \hat{Y}||$ minimizes, that means $\hat{Y} = b^T X$ solves the problem (A1). It is obvious that $\hat{Y}$ is defined by

$$\hat{Y} = \mathrm{Proj}_L Y = b^T X \text{ and } U = \hat{Y} - Y \in L^{\perp}.$$

Therefore $(Y - b^T X, X_i) = 0$ or $E(b^T X X_i) = E(X_i Y)$ for all $i = 1, \ldots, d$ or $b^T E(X^T X) = E(XY)$ which is the equation (A2). The rest of the above proposition is proved similarly.

**Acknowledgements.** This paper is based on the talk given at the Conference on Mathematics, Mechanics and Informatics, Hanoi, 7/10/2006, on the occasion of 50th Anniversary of Department of Mathematics, Mechanics and Informatics, Vietnam National University, Hanoi.

## References

[1] H. Follmer, M. Schweiser, Hedging of contingent claim under incomplete information, *App.Stochastic Analysis*, Edited by M.Davisand, R.Elliot, London, Gordan&Breach (1999) 389.
[2] H. Follmer, A. Schied, Stochastic Finance. *An introduction in discrete time*, Walter de Gruyter, Berlin- New York, 2004.
[3] J. Jacod, A.N. Shiryaev, Local martingales and the fundamental asset pricing theorem in the discrete case, *Finance Stochastic* **2**, pp. 259-272.
[4] M.J. Harrison, D.M. Kreps, Martingales and arbitrage in multiperiod securities markets, *J. of Economic Theory* 29 (1979) 381.
[5] M.J. Harrison, S.R. Pliska, *Martingales and stochastic integrals in theory of continuous trading, Stochastic Processes and their Applications* 11 (1981) 216.
[6] D. Lamberton, B. Lapayes, *Introduction to Stochastic Calculus Applied in Finance* , Chapman&Hall/CRC, 1996.
[7] M.L. Nechaev, *On mean -Variance hedging. Proceeding of Workshop on Math*, Institute Franco-Russe Liapunov, Ed. by A.Shiryaev, A. Sulem, Finance, May 18-19, 1998.
[8] Nguyen Van Huu, Tran Trong Nguyen, On a generalized Cox-Ross-Rubinstein option market model, *Acta Math. Vietnamica* 26 (2001) 187.
[9] M. Schweiser, Variance -optimal hedging in discrete time, *Mathematics of Operation Research* 20 (1995) 1.
[10] M. Schweiser, Approximation pricing and the variance-optimal martingale measure, *The Annals of Prob.* 24 (1996) 206.
[11] M. Schal, On quadratic cost criteria for option hedging, *Mathematics of Operation Research* 19 (1994) 131.