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SOME RESULTS CONCERNING A GRÄTZER'S PROHLEM

Nguyen Duc Dat
College of Natural Sciences - VNU

1. INTRODUCTION

Let L be an arbitrary lattice then $Sub(L)$ - a set of all sublattices of L, forms complete lattice. We say that $Sub(L)$ determines L up to isomorphism if: $Sub(L) \cong Sub(L') \Rightarrow L \cong L'$ for some lattice L', In [1] G. Grätzer has proposed the problem: "Find conditions under whch $Sub(L)$ determines L up to isomorphism".

Concerning the problem, in [2] N. D. Filippov has given:

Theorem (I). *Let L, L' be two arbitrary lattices, then $Sub(L) \cong Sub(L')$ iff there exits a square preserving bijection $\varphi : L \to L'$.*

H. M. Chuong in [3] also has proved:

Theorem (II). *Let M be a modular lattice of locally finite length which has no linear decompositions. Then $Sub(M)$ determines M up to isomorphism or dual isomorphism.*

Starting from Theorem (I) we will study the equivalence $\rho(\varphi)$ on L determined by a square preserving bijection: $\varphi : L \to L'$ and propose the concept of contractible sublattice. Using this concept we will generalize Theorem (II) to a larger class of lattices. The main result is

Theorem. *If the lattice L has no contractible sublattices then $Sub(L)$ determines L up to isomorphism or dual isomorphism.*

Finally, in Section 4 we will give an application of this theorem to the above mentioned problem.

2. CONCEPT OF CONTRACTIBLE SUBLATTICE

Consider a square preserving bijection: $\varphi : L \to L'$ where L, L' are arbitrary latices. The bijection φ induces a relation ρ_0 on L as follows:

$$a \,\rho_0\, b \text{ if either } a > b,\ \varphi(a) < \varphi(b) \text{ or } a < b,\ \varphi(a) > \varphi(b).$$

From ρ_0 we have the equivalence ρ:

$$a \,\rho\, b \text{ if either } a = b \text{ or } \exists x_0 = a,\ x_1, \ldots, x_{n-1},\ x_n = b : x_i \,\rho_0\, x_{i+1},\ i \in \{0, \ldots, n-1\}.$$

Definition 2.1. The equivalence on L generated by ρ_0 is called "φ determined" and denoted by $\rho(\varphi)$ or ρ.

Consider some examples:

In Fig. 1 we observe that $x_0 \rho x_3$ is determined by x_0, x_1, x_2, x_3 or only x_0, x_2, x_3 and $y_0 \rho y_4$ by $y_0, \ldots, y_4$ or only y_0, y_3, y_4.

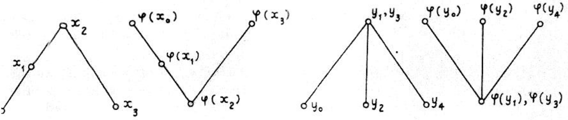

Fig. 1

If $x_0 \rho x_n$ is determined by $x_0, ..., x_n$ and $\exists i, j : o \leq i < j < n$ such that $x_i = x_j$ then , $x_1, ..., x_i, x_{j+1}, ..., x_n$ also determine $x_0 \rho x_n$. Thus we have:

efinition 2.2. Let $a \rho b$, the sequence $u = (x_0, ..., x_n)$ where $x_0 = a$, $x_n = b$ is called a sequence termining $a \rho b$ with the length $d(u) = n$ if:

(1) $x_i = x_j,\ i \neq j,\ i, j \in \{0, 1, ..., n\}$

(2) $x_i \rho x_{i+1},\ i \in \{0, 1, ..., n-1\}$

(3) $x_i > x_{i+1} \Leftrightarrow x_{i+1} < x_{i+2}$, $x_i < x_{i+1} \Leftrightarrow x_{i+1} > x_{i+2},\ i \in \{0, 1, ..., n-2\}$

Moreover, if u is also a chain then we say that u is a chain determining $a \rho b$.

For the sequences we have some lemmas, their proofs can be found in [4].

mma 2.3. *If $a \rho b$ is determined by the sequence $u = (a, x_1, x_2, b)$ then either u is a chain or ere exists v established from u such that v determines $a \rho b$ with $d(v) \leq 2$.*

mma 2.4. *Let $a \rho b$ and $a \neq b$ then there is a sequence u determining $a \rho b$ is one of the following o ways:*

1) u is a sequence with $d(u) = 2$.

2) u is a chain.

Now, we use the lemmas to prove the following theorem.

In what follows for short, it will be denoted by aSb or $a \parallel b$ when a is comparable or comparable with b respectively.

heorem 2.5. *Let $\varphi : L \to L'$ be a square preserving bijection and A with $|A| > 1$ be an uivalence class of $\rho(\varphi)$. Then:*

(a) A is a convex sublattice.

(b) If $\langle a, b ; c, d\rangle$ is a square on L then $c \in A \Leftrightarrow d \in A$.

oof.

1) Let $a, b \in A$. We can assume that $a \parallel b$. According to Lemma 2.4 there exists $x \in A$ such at (a, x, b) determines $a \rho b$. Without loss of generality we suppose that $x < a, b$, which implies $\leq a \wedge b$, i.e. $a \wedge b$, $a \vee b$ are equivalent to x. Thus A forms a sublattice.

To prove the convexity of A we take $a, b \in A$ and $z \in L$ such that $a < z < b$. We have to ove $z \in A$.

If $\varphi(z) < \varphi(a)$ or $\varphi(z) > \varphi(0)$ then obviously $z \in A$. Let us assume that $\varphi(a) < \varphi(z) < \varphi(b)$. ice $a, b \in A$ and aSb there exists a chain $u = (a, x_1, ..., x_{n-1}, b)$ determining $a \rho b$ (Lemma). We argue by induction on $d(u)$ in order to prove the implication $a < z < b$, $\varphi(a) < \varphi(z) <$ $b) \Rightarrow z \in A$.

a) $d = 2$, trivial.

b) Consider $u = (a, x_1, ... x_{n-1}, b)$ with $n > 2$. If $z \rho_0 x_1$ then $z \in A$, otherwise we compare a with x_1, it is easy to deduce that $x_1 < x < b$ and $\varphi(x_1) < \varphi(x) < \varphi(b)$. Since the chain $(x_1, \ldots x_{n-1}, b)$ determines $x_1 \rho b$ and has the length equal to $n-1$ we have the desired conclusion that $x \in A$.

2) Let $\langle a, b; c, d \rangle$ be a square in L with $c < d$, we prove that $c \in A \Rightarrow d \in A$ (symmetrically we have $d \in A \Rightarrow c \in A$). Let us assume that $\varphi(c) < \varphi(d)$. As $c \in A$ there exists $x \in A$ such that $c \rho_0 x$, From properties of the squares we have $d \rho_0 x$ i.e. $d \in A$, which was to be proved.

Now we consider the lattice L without attention to the square preserving bijection. Suppose that L has a proper sublattice A with $|A| > 1$, which satisfies the condition (a), (b) of 1.5. We define on L a congruence $\rho(A)$ such that A is one class and all the others consist of only one element. Thus the natural homomorphism $f : L \to L/\rho(A)$ identifies A with one element of the quotient lattice $L/\rho(A)$ and preserves all the squares which do not belong to A (Fig. 2). Therefore we have the following concept:

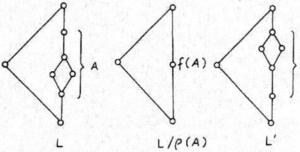

Fig. 2

Definition 2.6. Let L be an arbitrary lattice. The proper sublattice A with $|A| > 1$is called a contractible sublattice if:

(a) A is convex.

(b) If $\langle a, b; c, d \rangle$ is a square on L then $c \in A \Leftrightarrow d \in A$.

Note: 1) If L has at least one contractible sublattice then we can always establish a quare preserving bijection φ from L into another L' which is neither an isomorphism nor a dual isomorphism (Fig. 2).

2) The lattice K in Fig. 3 has no contractible sublattices and the square preserving bijections from K can only be an isomorphism or a dual isomorphism. This suggests us an idea to the main theorem in Section 3.

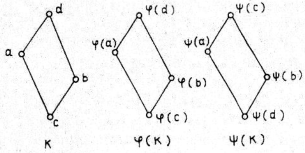

Fig. 3

3. MAIN THEOREM

Let $\varphi : L \to L'$ be a square preserving bijection. We introduce the concept of an invariable ;erval with respect to φ which we shall need in the sequel.

efinition 3.1. Let $u, v \in L$, $u < v$. If $\varphi(u) < \varphi(v)$ (or $\varphi(u) > \varphi(v)$) and $x \in [u, v] \Leftrightarrow \varphi(x) \in$ $(u), \varphi(v)]$ (or $\varphi(x) \in [\varphi(v), \varphi(u)]$) then $[u, v]$ is called an invariable interval of the type (I) (or I)) with respect to φ.

xample 3.2. Let $\langle a, b\,;\, c, d\rangle$ be a square with $c < d$, then $[c, d]$ is an invariable interval of the pe (I) of (II).

roof. Suppose $\varphi(c) < \varphi(d)$ we prove that $[c, d]$ is invariable of the type (I).

1) If $c < x < d$ then x is uncomparable with at least one of the two elements a, b. We can nsider $x \parallel a$, which follows $\varphi(x) \parallel \varphi(a)$. Thus, it is necessarily $\varphi(c) < \varphi(x) < \varphi(d)$.

2) Analogously, if $\varphi(c) < \varphi(x) < \varphi(d)$ then we can consider $\varphi(x) \parallel \varphi(a)$, therefore $x \parallel a$ and $< x < d$.

If $\varphi(c) > \varphi(d)$ we can also prove that $[c, d]$ is invariable of the type (II).

emma 3.3. *If $[u_i, v_i]$ $i = 1, 2$, are invariable intervals of the type (I) containing the subset $\neq \emptyset$ then $[u_1 \wedge u_2, v_1 \vee v_2]$ is also an invariable interval of the type (I) containing A.*

roof. We can always assume that $u_1 \parallel u_2$, $v_1 \parallel v_2$ and thus, we have the squares $\langle u_1, u_2\,;\, u_1 \wedge$ $_2, u_1 \vee u_2\rangle$, $\langle v_1, v_2\,;\, v_1 \wedge v_2, v_1 \vee v_2\rangle$ (Fig. 4).

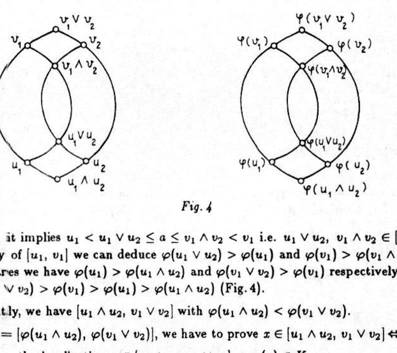

Fig. 4

For $a \in A$, it implies $u_1 < u_1 \vee u_2 \leq a \leq v_1 \wedge v_2 < v_1$ i.e. $u_1 \vee u_2$, $v_1 \wedge v_2 \in [u_1, v_1]$. Using ie invariability of $[u_1, v_1]$ we can deduce $\varphi(u_1 \vee u_2) > \varphi(u_1)$ and $\varphi(v_1) > \varphi(v_1 \wedge v_2)$. From the bove two squares we have $\varphi(u_1) > \varphi(u_1 \wedge u_2)$ and $\varphi(v_1 \vee v_2) > \varphi(v_1)$ respectively. That means, 'e obtain $\varphi(v_1 \vee v_2) > \varphi(v_1) > \varphi(u_1) > \varphi(u_1 \wedge u_2)$ (Fig. 4).

Consequently, we have $[u_1 \wedge u_2, v_1 \vee v_2]$ with $\varphi(u_1 \wedge u_2) < \varphi(v_1 \vee v_2)$.

Denote $K = [\varphi(u_1 \wedge u_2), \varphi(v_1 \vee v_2)]$, we have to prove $x \in [u_1 \wedge u_2, v_1 \vee v_2] \Leftrightarrow \varphi(x) \in K$.

(i) We prove the implication $x \in [u_1 \wedge u_2, v_1 \vee v_2] \Rightarrow \varphi(x) \in K$.

In the case that x is uncomparable with at least one of the two elements u_1, u_2 we have nmediately $\varphi(x) > \varphi(u_1 \wedge u_2)$. Comparing x with v_1, v_2 we also have $\varphi(x) < \varphi(v_1 \vee v_2)$.

In the case where $x\boldsymbol{S}u_1 x\boldsymbol{S}u_2$ simultaneously, there is $x > u_1$, u_2 and $\varphi(x) > \varphi(u_1)$, $\varphi(u_2)$ e. $\varphi(x) > \varphi(u_1 \wedge u_2)$. Further, considering the relation between x and v_1, v_2 we also obtain

$\varphi(x) < \varphi(v_1 \vee v_2)$.

(ii) Now, we prove: $\varphi(x) \in K \Rightarrow x \in [u_1 \wedge u_2, v_1 \vee v_2]$.

By contradiction we suppose that $x \notin [u_1 \wedge u_2, v_1 \vee v_2]$, then $x < u_1 \wedge u_2$ or $x > v_1 \vee v_2$. But this implies $\varphi(x) \notin K$, which is impossible.

The proof is completed.

Lemma 3.4. *If $[u_i, v_i]$, $i = 1, 2$, are invariable intervals of the type (II) containing the subset $A \neq \emptyset$ then $[u_1 \wedge u_2, v_1 \vee v_2]$ is also an invariable interval of the type (II) containing A.*

The proof is like that of Lemma 3.3.

Now, we consider the lattice L which has no contractible sublattices. Before provig the main result, let us present the following Lemma where 3.3 and 3.4 will be used.

Lemma 3.5. *Let L be a lattice having no contractible sublattices. If the square preservig bijection $\varphi : L \to L'$ is not isomorphic then $a < b \Leftrightarrow \varphi(a) > \varphi(b)$, $\forall a, b \in L$.*

Proof. Arguing by contradiction we assume $\exists a_1, a_2 \in L$ such that $a_1 < a_2$, $\varphi(a_1) < \varphi(a)$. Putting $A = \{a_1, a_2\}$, $A_1 = \{a_1\}$, $A_2 = \{a_2\}$, first we have to prove the following three assertons:

Assertion (A1): If there exists on L an invariable interval of the type (I) containingany of the three subsets A, A_1, A_2 then L has a contractible sublattice.

Proof of (A1). a) Let $[u, v]$ be an invariable interval of the type (I) containing A. Acording to Theorem 2.5 and since L has no contractible sublattices and φ is not isomorphic, L must be an equivalence class and thus $u \rho v$. Therefore $\exists x_1, x_2, ..., x_n \in L$ such that $u \rho_0 x_1$, $x_1 \rho_0 x_2 ...$, $x_n \rho_0 v$ (Lemma 2.4). Due to the symmetry we can assume $u < x_1$ ($\varphi(u) > \varphi(x_1)$). It is easy to deduce $v < x_1$ and hence we have $u < v < x_1$, $\varphi(x_1) < \varphi(u) < \varphi(v)$.

Consequently, we have $[u, v]$ as an invariable interval containing A with $v < x_1$. Denote M_1 as a set of all intervals similar to $[u, v]$. Put $M = \cup M_1$, then $A \subset M$ and $x_1 \notin M$.

Take $x, y \in M$ then $x \in [u_1, v_1]$, $y \in [u_2, v_2]$ $(\exists [u_i, v_i] \in M_1, i = 1, 2)$. Using Lemma 3.3 we have $[u_1 \wedge u_2, v_1 \vee v_2] \in M_1$ and $x \wedge y$, $x \vee y \in M$ M i.e. M is a sublattice of L.

It is easy to prove that M is contractible.

b) For the remain cases it will be enough to examine only the case where ther exists an invariable interval $[u, v]$ of the type (I) which contains A_1 and does not contain A_2. Hre we take M_1 as family of all invariable intervals like $[u, v]$ and $M = \cup M_1$. We can also prove the ontraction of M.

Thus (A1) is proved.

Assertion (A2): If there exist on L an invariable interval of the type (II) containing A_i and which does not contain A_j, $i, j = 1, 2$, $i \neq j$, then L has a contractible sublattice.

Proof of (A2). By similar arguments as in (A1) part b) and using Lemma 3.4 we ome to the desired conclusion.

Assertion (A3): If there exist on L neither invariable intervals as in (A1) nor in (A2) hen either $[a_1, a_2]$ or (a_1, a_2) (open interval) is a contratible sublattice.

Proof of (A3). Denote $X = [a_1, a_2]$, $Y = (a_1, a_2)$.

(i) If $X = L$ we examine Y. As L has no contractible sublattices, it is clear tht $|Y| > 1$ Evidently Y is a convex sublattice. Now we verify condition (b) (Definition 2.6). Let $(a, b; c, d)$

a square on L with $c < d$. Due to the symmetry we only have to show $c \in Y \Rightarrow d \in Y$. contradiction we assume $d \notin Y$. It is necessarily $d = a_2$, but $[c, d]$ is an invariable interval xample 3.2) and $a_1 \notin [c, d]$. This contradicts the assumptions.

(ii) If $X \neq L$ we show that X is contractible. We have only verify condition (b). Consider e square $\langle a, b\,;\, c, d\rangle$ on L where $c < d$. We only have to prove the implication $c \in X \Rightarrow d \in X$. ssume that $d \notin X$. Comparing d with a_2 we have:

If $d \parallel a_2$ then a_2 belongs to $[d \wedge a_2,\, d \vee a_2]$ which is an invariable interval (Example 3.2). urthermore, if it is of the type (II) then it dose not contain A_1. This contradicts the assumptions.

If $d \mathcal{S} a_2$ then $d > a_2$ and thus $a_2 \in [c, d]$. But $[c, d]$ is invariable, that is impossible. Thus, it necessarily $d \in X$ and (A3) is proved.

Now we return to the proof of Lemma 3.5. The assertions (A1), (A2), (A3) show that L lways has a contractible sublattice. This is our desired contradiction.

We are now in a position to formulate the main theorem.

heorem 3.6. *If the lattice L has no contractible sublattices then $Sub(L)$ determines L up to omorphism or dual isomorphism.*

roof. Let $f : Sub(L) \to Sub(L')$ be a lattice isomorphism for some L'. We have to prove either $\cong L'$ or $L \overset{\bullet}{\cong} L'$ (dually isomorphic).

According to Theorem (I), f induces a square preserving bijection $\varphi : L \to L'$. Using Theorem .5 and the fact that L has no contractible sublattices we have only the following cases:

1) If all equivalence classes of $\rho(\varphi)$ consist of only one element then φ is an isomorphism.

2) If L is the only equivalence class of $\rho(\varphi)$ then φ is not isomorphic, in this case φ must be dual isomorphism due to Lemma 3.5.

The proof is completed.

4. APPLICATION

First, we notice that for a contractible sublattice A of L, it is easy to deduce the following roperty:

(P) Let $k \in L \setminus A$ and $a \in A$. If $k < a$ then $k < x$, $\forall x \in A$.

Now we consider the lattice $\sum$ of all topologies of a given set $X \neq \emptyset$ (see [5]), where the zero-lement is the topology $O = \{\emptyset, X\}$ and the atoms are all the topologies of the form $\{\emptyset, A, X\}$ with $\emptyset \neq A \neq X$. Indicating $\{t_i : i \in I\}$ as a set of atoms we have: $\forall t \in \sum$, $t \neq 0$, $t = \vee(t_j,\, j \in J)$ or some subset $\emptyset \neq J \subset I$.

tatement 4.1. *The lattice $\sum$ has no contractible sublattices.*

Proof. We argue by contradiction. Assume that $\sum$ has a contractible sublattice K. As $|K| > 1$ we an take t, $t' \in K$ such that $t < t'$. Thus $t' > 0$ and $t' = \vee(t_j,\, j \in J)$ for some non-empty subset $J \subset I$. According to (P) $t_j < t$, $j \in J$. This implies that $t' = t$ which is impossible.

In short $\sum$ has no contractible sublattices, which was to be proved.

Note: In another paper we will also show the other lattices which have no contractible sublattices. They are the modular and semi - modular lattices having no linear decompositions, the free lattices etc. This means that the class of lattices satisfying the conditions of Theorem 3.6 is large enough,

and it contains the class of lattices mentioned in Theorem (II).

Now there is a natural question that what can be done with the lattices that have contractible sublattices.

First we consider the contractible sublattices A, B in the examples in Fig. 5.

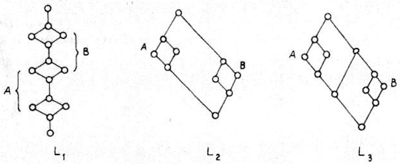

Fig. 5

In Fig. 5 it is observed that L_1 is linearly decompossable, A, B are the maximal cotractible sublattices and $A \cap B \neq \emptyset$. And L_2, L_3 have no linear decompositions where A, B are alsomaximal contractible sublattices, but $A \cap B = \emptyset$.

In general, we have the assertion:

(M) If L has no linear decompositions, and A, B are the different maximal cotractible sublattices of L then $A \cap B = \emptyset$.

Now we consider the lattice L for which every contractible sublattices is embeddd into a maximal one and let A_i, $i \in I$, be all the maximal contractible sublattices in L. Due t (M) we can define a congruence $\rho(I)$ on L as follows: every A_i, $i \in I$ is one class and the othr classes consist of only one element. Thus, the quotient lattice $L/\rho(I)$ has no contractible sublatices.

For brevety, we say that the condition (G) holds for the lattice L if $Sub(L)$ determies L up to isomorphism. Due to Theorem 3.6 and the fact that $L/\rho(I)$ has no contractible sublatices, the following result is easily obtained:

Let L be a lattice having no linear decompositions and every contractible sublatticeof which is embedded into a maximal one, let $\{A_i,\ i \in I\}$ be the family of all the maximal corractible sublattices. If A_i, $i \in I$ satisfy condition (G) then L is determined by $Sub(L)$ up to isororphism or dual isomorphism.

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MỘT SỐ KẾT QUẢ CHO BÀI TOÁN GRÄTZER

Nguyễn Đức Đạt
Đại học Khoa học tự nhiên - ĐHQGHN

Trong [1], G. Grätzer đã nêu bài toán: "Tìm điều kiện trên dàn L sao cho $Sub(L)$ xác định L i khác nhau một đẳng cấu".

Theo hướng của Hoàng Minh Chương [3], chúng tôi nghiên cứu các điều kiện trên một dàn sao cho L được xác định bởi $Sub(L)$ sai khác nhau một đẳng cấu hoặc đối đẳng cấu.

Kết quả của N. D. Filippov [2] đã gợi ra ý tưởng cần phải bắt đầu bằng việc nghiên cứu các ng ánh bảo toàn hình thoi từ L lên một dàn L' nào đó. Do vậy chúng tôi đi đến khái niệm dàn n co được và chứng minh kết quả sau: Nếu dàn L không có dàn con co được thì $Sub(L)$ xác nh L sai khác nhau một đẳng cấu hoặc một đối đẳng cấu. Phần áp dụng chỉ ra nhiều kiểu dàn uộc vào các dàn không có dàn con co được. Ngoài lớp các dàn này chúng tôi còn xét một số n có dàn con co được.

Như vậy, bằng khái niệm dàn con co được chúng tôi đã đưa ra một lời giải thú vị cho bài án Grätzer.